A NEW APPROACH TO ORDER POLYNOMIALS OF LABELED POSETS AND THEIR GENERALIZATIONS

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Abstract. In this paper, we first give formulas for the order polynomial $\Omega(P,\omega;t)$ and the Eulerian polynomial $e(P,\omega;\lambda)$ of a finite labeled poset (P,ω) using the adjacency matrix of what we call the ω-graph of (P,ω). We then derive various recursion formulas for $\Omega(P,\omega;t)$ and $e(P,\omega;\lambda)$ and discuss some applications of these formulas to Bernoulli numbers and Bernoulli polynomials. Finally, we give a recursive algorithm using a single linear operator on a vector space. This algorithm provides a uniform method to construct a family of new invariants for labeled posets (P,ω), which includes the order polynomial $\Omega(P,\omega;t)$ and the invariant $\tilde{e}(P,\omega;\lambda)=\frac{e(P,\omega;\lambda)}{(1-\lambda)^{|P|+1}}$. The well-known quasi-symmetric function invariant of labeled posets and a further generalization of our construction are also discussed.

Contents

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1. **Introduction**

In this paper, we study order polynomials $\Omega(P, \omega; t)$ and Eulerian polynomials $e(P, \omega; \lambda)$ of labeled posets (P, ω) (see [St1] and [St2]) and their generalizations. Motivated by the way that strict order polynomials $\bar{\Omega}(T)$ of rooted trees appear as coefficients in the tree expansions of formal flows of analytic maps of $\mathbb{C}^n$ in [WZ], we first give formulas in Section 2 for the order polynomial $\Omega(P, \omega; t)$ and the Eulerian polynomial $e(P, \omega; \lambda)$ in terms of the adjacency matrix $A(P, \omega)$ of what we call the ω-graph of a labeled poset (P, ω). We also derive various recursion formulas for $\Omega(P, \omega; t)$ and $e(P, \omega; \lambda)$. Some applications of these formulas to Bernoulli numbers and Bernoulli polynomials are discussed in Section 2.3. Although some results in Section 2 can be found in literature, it seems to us that there is no source that treats order polynomials and Eulerian polynomials of labeled posets in the manner presented here. In Section 3, motivated by the recursion formulas derived in Section 2 and also the construction of a family of invariants of rooted trees in [Z], we construct a family of invariants for labeled posets by a recursive algorithm (see Algorithm 3.1) in terms of a single linear operator Ξ on a vector space A. We then show that the order polynomial $\Omega(P, \omega; t)$ and the invariant $\tilde{e}(P, \omega; \lambda) = \frac{e(P,\omega;\lambda)}{(1-\lambda)^{|P|+1}}$ of a labeled poset (P, ω) belong to this family of invariants. In Section 3.3, we consider the quasi-symmetric function invariant $K(P, \omega; x)$ (see [G], [MR] and [St3]) of a labeled poset (P, ω). Even though $K(P, \omega; x)$ can not be recovered from our general construction, we give a recursion formula for $K(P, \omega; x)$ (see Proposition 3.6) which suggests a further generalization of our construction. Finally, we consider invariants of unlabeled posets P derived from Algorithm 3.1 by identifying unlabeled posets with labeled posets in certain ways.

2. **Order Polynomials of Labeled Posets**

Once and for all, we fix the following notation and conventions.

Notation and Conventions:

(1) We denote by $\mathbb{N}$ (resp. $\mathbb{N}^+$) the set of all non-negative (resp. positive) integers. For any $n \in \mathbb{N}^+$, the totally ordered poset $\{1, 2, \cdots, n\}$ will be denoted by $[n]$.

(2) For any $n \in \mathbb{N}$, we denote by $\mathbb{P}_n$ the set of posets with n elements. We set $\mathbb{P} = \bigsqcup_{n\in\mathbb{N}} \mathbb{P}_n$. The empty set $\emptyset$ will always be treated as a finite poset and will be viewed as an ideal of any poset P. The poset with a single element is called the *singleton* and is denoted by $\bullet$.

(3) For any poset $P \in \mathbb{P}$, we denote by $V(P)$ the set of elements of P, and by $|P|$ the cardinality of $V(P)$. We denote by $L(P)$ the set of minimum elements of P.
(4) For any $P \in \mathbb{P}$, we denote by $\mathcal{I}(P)$ the set of all ideals including $\emptyset$ and P itself.
(5) For any $n \in \mathbb{N}$, we denoted by A_n the anti-chain with n elements. The empty poset $\emptyset$ and the singleton $\bullet$ are anti-chains.
(6) We say a poset P is a *rooted tree* if its Hasse diagram is a rooted tree with the root being the unique minimum element.
(7) For any $n \in \mathbb{N}$, we let S_n be the *shrub* with $n+1$ vertices. It is the poset in $\mathbb{P}_{n+1}$ whose Hasse diagram is the unique rooted tree of height ≤ 1. (Equality holds unless $n = 0$.) Note that $S_0 = \bullet$, the singleton.
(8) For a finite poset P, we set the following notation:
   (a) $S \trianglelefteq P$: S is an ideal of P.
   (b) $S \triangleleft P$: $S \trianglelefteq P$ but $S \neq P$.

In this paper "poset" will always mean a partially ordered finite set. Recall that a labeled poset (P, ω) is a poset P with an injective map $\omega : V(P) \to \mathbb{N}^+$. When ω as a map from P to $\mathbb{N}^+$ is order-preserving (resp. order-reversing), we say (P, ω) is a *naturally* (resp. *strictly*) *labeled* poset. We say two labeled posets (P, ω) and (Q, ϕ) are *equivalent* if there is an isomorphism $f : P \to Q$ of posets and an order-preserving bijection $g : \mathrm{Im}(\omega) \to \mathrm{Im}(\phi)$ such that $g \circ \omega = \phi \circ f$. Note that equivalent posets have same order polynomial. In this paper, the term "labeled poset" will always refer to an equivalence class. In other words, any two equivalent labeled posets are considered to be the same poset. Note that each equivalence class has a unique representative (P, ω) with $\omega(P) = [|P|]$.

Given (P, ω) a labeled poset and S a subset of $V(P)$, (S, ω) will denote the induced labeled sub-poset on S. We say that S is *ω-natural* if the restriction $\omega : S \to \mathbb{N}^+$ is order-preserving. We denote by $\mathcal{N}_\omega(P)$ the set of all ω-natural subsets of $V(P)$ and $\mathcal{I}_\omega(P)$ the set of all ω-natural ideals of (P, ω). For convenience, we will always treat the empty set $\emptyset$ as a ω-natural ideal of (P, ω).

Note that, for any finite poset P, order-preserving labelings and order-reversing labelings for P always exist.

Now, for any labeled poset (P, ω), we define a directed graph $\mathcal{G}(P, \omega)$ as follows.

The set of vertices of $\mathcal{G}(P, \omega)$ is the set of ideals of P, i.e. elements of $\mathcal{I}(P)$. Two elements $I, J \in \mathcal{I}(P)$ are connected by an arrow pointing to I if and only if $I \subsetneq J$ and $J \backslash I \in \mathcal{N}_\omega(P)$.

We call the directed graph $\mathcal{G}(P,\omega)$ the *ω-graph* of the labeled poset (P,ω). We let $A(P,\omega)$ denote the adjacency matrix of the ω-graph $\mathcal{G}(P,\omega)$. Note that we can always arrange the ideals of P so that the matrix $A(P,\omega)$ is strictly upper-triangular. In particular, the matrix $A(P,\omega)$ is always nilpotent. In this paper we will always assume that the matrix $A(P,\omega)$ is strictly upper-triangular.

Recall that a (directed) *path* of length k in a directed graph D is a list $v_0,\dots,v_k$ of vertices of D such that (v_i,v_{i+1}) is an arc (i.e., an edge directed toward to v_{i+1}) in D for $0\le i<k$. Define a (directed) *multi-path* of length k in D to be a list $v_0,\dots,v_k$ of vertices such that, for $0\le i<k$, either (v_i,v_{i+1}) is an arc in D or $v_{i+1}=v_i$. For each $k\in\mathbb{N}^+$, we denote by $\mathcal{A}_k(P,\omega)$ the set of all directed multi-paths of length k from $\mathcal{G}(P,\omega)$ connecting the elements $\emptyset$ and P and $\mathcal{C}_k(P,\omega)$ the set of all directed paths of length k of $\mathcal{G}(P,\omega)$ connecting the elements $\emptyset$ and P. We let $a_k(P,\omega)$ and $c_k(P,\omega)$ the cardinalities of $\mathcal{A}_k(P,\omega)$ and $\mathcal{C}_k(P,\omega)$, respectively.

2.1. **Order Polynomials of Labeled Posets.** Let (P,ω) be a labeled poset. Recall that a map $f:P\to[n]$ with $n\in\mathbb{N}^+$ is said to be *ω-order-preserving* if f is order-preserving and, for any $x>y$ in P with $\omega(x)<\omega(y)$, we have $f(x)>f(y)$.

The following two lemmas follow immediately from the definitions.

Lemma 2.1. *Let (P,ω) be a labeled poset.*

(a) If ω is a natural labeling, i.e. $\omega:P\to\mathbb{N}^+$ is order-preserving, then a map $\varphi:P\to[n]$ is ω-order-preserving if any only of it is order-preserving. In this case, any subset of P is ω-natural.

(b) If $\omega:P\to\mathbb{N}^+$ is order-reversing, then a map $\varphi:P\to[n]$ is ω-order-preserving if and only if it is strictly order-preserving. A subset $S\subset P$ is ω-natural if and only of S with induced poset structure from P is an anti-chain.

Lemma 2.2. *An order-preserving map $f:P\to[n]$ is ω-order-preserving if and only if $f^{-1}(k)$ is ω-natural for any $k\in[n]$.*

Lemma 2.3. *For any $n\in\mathbb{N}^+$, the set of ω-order-preserving maps $f:P\to[n]$ is in one-to-one correspondence with the set $\mathcal{A}_n(P,\omega)$. Hence, the number of ω-order-preserving maps $f:P\to[n]$ equals $a_n(P,\omega)$.*

Proof: For any ω-order-preserving map $f:P\to[n]$, Set $I_k=\{x\in P|f(x)\le k\}$ for any $1\le k\le n$. It is easy to see that $\emptyset\prec I_1\prec\cdots\prec I_n=P$ is a multi-path of the directed graph $\mathcal{G}(P,\omega)$. Conversely, for any multi-path $\emptyset\prec I_1\prec\cdots\prec I_n=P$ of $\mathcal{G}(P,\omega)$, we define a

map $f : P \to [n]$ by setting $f(I_1) = 1$ and $f(I_k \backslash I_{k-1}) = k$ for any $2 \leq k \leq n$. It is easy to check that the map f is ω-order-preserving and the correspondences defined above are inverse to each other. $\square$

Now, for any labeled poset (P, ω), we define a function $\theta(P, \omega; \cdot\,) : \mathbb{N}^+ \to \mathbb{N}$ by setting $\theta(P, \omega; n)$ $(n \in \mathbb{N}^+)$ to be the number of ω-order-preserving maps $f : P \to [n]$. From Lemma 2.3, we see that $\theta(P, \omega; n) = a_n(P, \omega)$ for any $n \in \mathbb{N}^+$.

Next we give a proof of the following well-known theorem.

Theorem-Definition 2.4. *For any labeled poset (P, ω), there exists a unique polynomial $\Omega(P, \omega; t)$ such that $\Omega(P, \omega, n) = \theta(P, \omega; n)$ for any $n \in \mathbb{N}^+$.*

Note that $\Omega(\emptyset, t) = 1$.

To prove this theorem, we first define the following $|\mathcal{I}(P)| \times |\mathcal{I}(P)|$ matrices.

$$\Phi(P, \omega) := \ln(1 + A(P, \omega)) = \sum_{n=1}^{\infty} \frac{(-1)^n A^{n-1}(P, \omega)}{n}, \tag{2.1}$$

$$\Theta(P, \omega; t) := e^{t\Phi(P,\omega)} = \sum_{n=0}^{\infty} \frac{\Phi^n(P, \omega)}{n!} t^n, \tag{2.2}$$

where t is a formal variable.

First note that $\Phi(P, \omega)$ is a strict upper triangular matrix, for the adjacency matrix $A(P, \omega)$ is. In particular, $\Phi(P, \omega)$ is nilpotent. Hence $\Theta(P, \omega, t)$ is a upper triangular matrix with 1 on the diagonal and each entry in the polynomial algebra $\mathbb{Q}[t]$.

The theorem above follows immediately from the following lemma.

Lemma 2.5. *For any $I, I' \in \mathcal{I}(P)$, let $\theta(I, I'; t)$ denote the $(I, I')^{th}$ entry of the matrix $\Theta(P, \omega; t)$. Then,*

$$\theta(I, I'; t) = \begin{cases} 0 & \text{if } I \nsubseteq I'; \\ \Omega(I' \backslash I, \omega, t) & \text{if } I \subseteq I'. \end{cases} \tag{2.3}$$

In particular, $\theta(\emptyset, P; t)$ is the same as the order polynomial $\Omega(P, \omega; t)$.

Proof: For any $n \in \mathbb{N}^+$, we have

$$\begin{aligned} \Theta(P, \omega; n) &= e^{n\Phi(P,\omega)} \\ &= e^{n \ln(1 + A(P,\omega))} \\ &= (1 + A(P, \omega))^n. \end{aligned}$$

By a standard result in algebraic graph theory (see [B]), the $(I,I')^{th}$ entry of the matrix $(1+A(P,\omega))^n$ equals to the number of directed multi-paths in the ω-graph $\mathcal{G}(P,\omega)$ connecting I and I'. It is easy to see that it is also same as the number of the directed multi-paths in the ω-graph $\mathcal{G}(I'\backslash I,\omega)$ of the induced labeled sub-poset $(I'\backslash I,\omega)$ connecting $\emptyset$ and $I'\backslash I$ when $I\subseteq I'$. Therefore, for any $n\in\mathbb{N}^+$, we have $\theta(I,I';n)=0$ if $I\nsubseteq I'$ and $\theta(I,I';n)=\theta(I'\backslash I,\omega;n)$ if $I\nsubseteq I'$. By Lemma 2.3 and the definition of order polynomials, we see that Eq. (2.3) holds. $\square$

Next, we derive some properties of order polynomials of labeled posets (P,ω) and also study the combinatorial interpretation of the numerical invariants defined by the matrix $\Phi(P,\omega)$.

Theorem 2.6. *For any labeled poset (P,ω) and $s,t\in\mathbb{C}$, we have*

$$\Omega(P,\omega;s+t)=\sum_{S\in\mathcal{I}(P)}\Omega(S,\omega;s)\Omega(P\backslash S,\omega;t). \tag{2.4}$$

Proof: First, from Eq.(2.1) we have

$$\Theta(P,\omega;s+t)=\Theta(P,\omega;s)\Theta(P,\omega;t). \tag{2.5}$$

By Lemma 2.3 and the equation above, we have

$$\begin{aligned}\Omega(P,\omega;s+t) &= \theta(\emptyset,P,s+t)\\ &= \sum_{S\in\mathcal{I}(P)}\theta(\emptyset,S;s)\theta(S,P;t)\\ &= \sum_{S\in\mathcal{I}(P)}\Omega(S,\omega;s)\Omega(P\backslash S,\omega;t).\end{aligned}$$

$\square$

The next theorem is a direct consequence of Theorem 2.6 above, but is more convenient and useful. It not only provides an algorithm for the calculation of order polynomials of labeled posets, but also is the main motivation for the further generalizations that will be discussed in Section 3.

First we denote by Δ the difference operator on the polynomial algebra $\mathbb{C}[t]$, i.e.

$$(\Delta f)(t)=f(t+1)-f(t).$$

for any $f \in \mathbb{C}[t]$. We also define the linear operator $\Delta^{-1} : \mathbb{C}[t] \to \mathbb{C}[t]$ by setting $(\Delta^{-1} f)(t)$ to be the unique polynomial $g(t)$ such that

$$\begin{aligned} g(0) &= 0, \\ (\Delta g)(t) &= f(t). \end{aligned}$$

Theorem 2.7. *For any labeled poset (P, ω) with $|P| = p \geq 2$, we have*

$$\Omega(P, \omega; t) = \Delta^{-1} \sum_{\emptyset \neq S \in \mathfrak{I}_\omega(P)} \Omega(P \backslash S, \omega; t). \tag{2.6}$$

Note that the sum in Eq. (2.6) runs over a (usually) smaller set, namely the set $\mathfrak{I}_\omega(P)$ of ω-natural ideals of (P, ω), than the one in Eq. (2.4), which runs over the set $\mathfrak{I}(P)$ of all ideals of P.

Proof: From Lemma 2.2, for any labeled poset (Q, η), we have

$$\Omega(Q, \eta; 1) = \begin{cases} 1 & \text{if } (Q, \eta) \text{ is natural;} \\ 0 & \text{otherwise.} \end{cases}$$

This and Eq. (2.4), setting $s = 1$, yield

$$\Delta\Omega(P, \omega; t) = \sum_{\emptyset \neq S \in \mathfrak{I}_\omega(P)} \Omega(P \backslash S, \omega; t). \tag{2.7}$$

Since $\Omega(P, \omega; t)$ has no constant term, Eq. (2.6) follows immediately from the equation above. $\square$

Finally, to end this subsection, let us look at a combinatorial interpretation of coefficients of order polynomials of labeled posets.

First, from Eq. (2.2), we have

$$\frac{d}{dt} \Theta(P, \omega; 0) = \Phi(P, \omega). \tag{2.8}$$

For any $I, I' \in \mathfrak{I}(P, \omega)$, we define $\phi_{I',I} \in \mathbb{Q}$ by writing $\Phi(P, \omega) = (\phi_{I',I})_{I,I' \in \mathfrak{I}(P)}$. From Lemma 2.5 and Eq. (2.8), we see that $\phi_{I',I}$ is equal to the coefficient of t in the order polynomial $\Omega(I' \backslash I, \omega; t)$ if $I \subsetneq I'$ and 0 otherwise. In particular, the quantity $\phi_{I',I}$ depends only on the induced labeled sub-poset structure on the subset $I' \backslash I$ from (P, ω), not on the ambient labeled poset (P, ω). This also follows from Eq. (2.9) below. So, from now on, $\phi_{I',I}$ will also be denoted by $\phi(I' \backslash I; \omega)$ when $I \subsetneq I'$. Hence, for any labeled poset (P, ω), we have a unique well defined rational number $\phi(P, \omega)$.

From Eq. (2.1), it is easy to see that the following lemma is true.

Lemma 2.8. *For any labeled poset (P, ω) and $I, I' \in \mathcal{I}(P, \omega)$, we have*

$$\phi_{I,I'} = \begin{cases} \sum_{k=1}^{|I'\backslash I|} (-1)^{k-1} \frac{c_k(I'\backslash I, \omega)}{k} & \text{if } I \subsetneq I'; \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

In particular, for any non-empty labeled poset (P, ω), we have

$$\phi(P, \omega) = \sum_{k=1}^{|P|} (-1)^{k-1} \frac{c_k(P, \omega)}{k}. \tag{2.10}$$

Note that $c_k(I'\backslash I, \omega)$ is the number of directed chains of the ω-graph $\mathcal{G}(P, \omega)$ connecting I and I'. It is also equal to the number of directed chains of the ω-graph $\mathcal{G}(I'\backslash I, \omega)$ of the induced labeled sub-poset $(I'\backslash I, \omega)$ of (P, ω) connecting $\emptyset$ and $I'\backslash I$.

Interestingly, the sequence of rational numbers $\{\phi(P, \omega)\}$ indexed by the set of labeled posets satisfies the following recursion formula.

Proposition 2.9. (1) $\phi(\emptyset) = 0$ *and* $\phi(\bullet) = 1$.

(2) *For any non-empty poset P, set $n_{(P,\omega)}$ equal to* 1 *if P is ω-natural and* 0 *otherwise. Then*

$$\begin{aligned} \phi(P, \omega) = {} & n_{(P,\omega)} \\ & - \sum_{r=2}^{|P|} \frac{1}{r!} \sum_{\emptyset \neq I_1 \lhd I_2 \lhd \cdots \lhd I_r = P} \phi(I_1, \omega)\phi(I_2\backslash I_1, \omega) \cdots \phi(I_r\backslash I_{r-1}, \omega), \end{aligned} \tag{2.11}$$

or equivalently,

$$\begin{aligned} \sum_{r=1}^{|P|} \frac{1}{r!} \sum_{\emptyset \neq I_1 \lhd I_2 \lhd \cdots \lhd I_r = P} & \phi(I_1, \omega)\phi(I_2\backslash I_1, \omega) \cdots \phi(I_r\backslash I_{r-1}, \omega) \\ & = n_{(P,\omega)}. \end{aligned} \tag{2.12}$$

Proof: (1) is obvious from the definition of ϕ and Eq. (2.10). To prove (2), first from Eq. (2.1), we have

$$e^{\Phi(P,\omega)} = 1 + A(P, \omega).$$

Hence we have,

$$\sum_{r=1}^{\infty} \frac{1}{r!} \Phi^r(P, \omega) = A(P, \omega).$$

Note that $\Phi^r(P,\omega)=0$ for any $r>|P|$, since there are no paths of length greater than $|P|$ in the ω-graph $\mathcal{G}(P,\omega)$. Hence, we have

$$\sum_{r=1}^{|P|}\frac{1}{r!}\sum_{I_1,I_2,\cdots,I_r=P\in\mathcal{I}(P)}\phi(\emptyset,I_1)\phi(I_1,I_2)\cdots\phi(I_{r-1},P)=n_{(P,\omega)}.$$

Then, Eq. (2.12) follows from (1) and Lemma 2.8. □

From Eq. (2.2) and Lemma 2.5, it is easy to see that we can write down the order polynomials $\Omega(P,\omega;t)$ in terms of the rational numbers ϕ as follows.

Proposition 2.10. *For any non-empty labeled poset (P,ω), we have*

$$\Omega(P,\omega;t)=\sum_{r=1}^{|P|}\left(\frac{1}{r!}\sum_{\emptyset\neq I_1\lhd I_2\lhd\cdots\lhd I_r=P}\phi(I_1,\omega)\phi(I_2\backslash I_1,\omega)\cdots\phi(I_r\backslash I_{r-1},\omega)\right)t^r.$$

In terms of the ϕ's, we also have the following recursion formula for order polynomials of labeled posets.

Proposition 2.11. *For any finite poset P, we have*

$$\begin{aligned}\Omega'(P,\omega;t) &= \sum_{S\in\mathcal{J}(P)}\phi(S,\omega)\Omega(P\backslash S,\omega;t)\\ &= \sum_{S\in\mathcal{J}(P)}\phi(P\backslash S,\omega)\Omega(S,\omega;t),\end{aligned}\tag{2.13}$$

or in other words,

$$\begin{aligned}\Omega'(P,\omega;t) &= \sum_{S\in\mathcal{J}(P)}\Omega'(S,\omega;0)\Omega(P\backslash S,\omega;t)\\ &= \sum_{S\in\mathcal{J}(P)}\Omega'(P\backslash S,\omega;0)\Omega'(S,\omega;t).\end{aligned}\tag{2.14}$$

Proof: By applying $\frac{d}{dt}$ to Eq. (2.2), we get

$$\frac{d}{dt}\Theta(P,\omega;t)=\Phi(P,\omega)\Theta(P,\omega;t)=\Theta(P,\omega;t)\Phi(P,\omega).\tag{2.15}$$

Then the equations in the theorem follow immediately from the equation above and Lemma 2.5. □

2.2. **Eulerian Polynomials of Labeled Posets.** In this subsection, we give a formula for the Eulerian polynomial $e(P,\omega;\lambda)$ of a labeled poset (P,ω) in terms of its adjacency matrix $A(P,\omega)$. We also derive certain recursion formulas for the Eulerian polynomial $e(P,\omega;\lambda)$.

Let (P,ω) be a labeled poset. Recall that the Eulerian polynomial $e(P,\omega;\lambda)$ of (P,ω) is defined by the equation

$$\sum_{n=0}^{\infty}\Omega(P,\omega;n)\lambda^n=\frac{e(P,\omega;\lambda)}{(1-\lambda)^{|P|+1}}. \tag{2.16}$$

Remark 2.12. *It is a standard fact in combinatorics that $e(P,\omega;\lambda)$ defined above is indeed a polynomial; however, this will be clear from Eq. (2.23) below.*

We set

$$\tilde{e}(P,\omega;\lambda)=\frac{e(P,\omega;\lambda)}{(1-\lambda)^{|P|+1}} \tag{2.17}$$

and

$$E(P,\omega;\lambda):=\sum_{n=0}^{\infty}\Theta(P,\omega;n)\lambda^n. \tag{2.18}$$

and, for any $I,I'\in\mathfrak{I}(P)$, define $\tilde{e}(I,I';\lambda)$ by writing $E(P,\omega;\lambda)=(\tilde{e}(I,I';\lambda))_{I,I'\in\mathfrak{I}(P)}$.

From Lemma 2.5 and Eq. (2.16) and (2.18), it is easy to see that we have the following lemma.

Lemma 2.13. *For any $I,I'\in\mathfrak{I}(P)$, we have*

$$\tilde{e}(I,I';\lambda)=\begin{cases}\tilde{e}(I'\backslash I,\omega;\lambda) & \text{if } I\subseteq I';\\ 0 & \text{otherwise.}\end{cases} \tag{2.19}$$

In particular, $\tilde{e}(I,I';\lambda)$ depends only on the reduced labeled sub-poset structure on the subset $I'\backslash I$ of P, not on the ambient labeled poset P. So, from now on, we will also denote $\tilde{e}(I,I';\lambda)$ by $\tilde{e}(I'\backslash I,\omega;\lambda)$ when $I\subseteq I'$.

Theorem 2.14. *For any labeled poset (P,ω), we set $\widetilde{A}(P,\omega):=I+A(P,\omega)$. Then, we have*

$$E(P,\omega;\lambda)=(I-\lambda\widetilde{A}(P,\omega))^{-1} \tag{2.20}$$

and

$$E(P,\omega;\lambda)=\frac{1}{1-\lambda}\left(I-\frac{\lambda}{1-\lambda}A(P,\omega)\right)^{-1}. \tag{2.21}$$

Proof: It is easy to see that Eq. (2.21) is a direct consequence of Eq. (2.20). To prove Eq. (2.20), first by Eq. (2.2) and (2.18), we have

$$\begin{aligned} E(P,\omega;\lambda) &= \sum_{n=0}^{\infty} \Theta(P,\omega;n)\lambda^n \\ &= \sum_{n=0}^{\infty} (\mathrm{I} + A(P,\omega))^n \lambda^n \\ &= \sum_{n=0}^{\infty} \widetilde{A}(P,\omega)^n \lambda^n \\ &= (I - \lambda \widetilde{A}(P,\omega))^{-1}. \end{aligned}$$

□

Corollary 2.15. *For any non-empty labeled poset* (P,ω)*, we have*

$$\begin{aligned} \tilde{e}(P,\omega;\lambda) &= \sum_{k=1}^{|P|} a_k(P,\omega)\lambda^k \\ &= \frac{1}{1-\lambda}\sum_{k=1}^{|P|} c_k(P,\omega)\left(\frac{\lambda}{1-\lambda}\right)^k \end{aligned} \tag{2.22}$$

and

$$e(P,\omega;\lambda) = \sum_{k=1}^{|P|} c_k(P,\omega)\lambda^k(1-\lambda)^{|P|-k}. \tag{2.23}$$

Proof: Eq. (2.22) follows from Eq. (2.20), (2.21) and (2.19); while Eq. (2.23) follows from Eq. (2.22) and (2.17). □.

Recall that, for any non-empty directed graph G, the *chain polynomial* $c(G;\mu)$ of G is defined to be

$$c(G,\mu) = \sum_{k=0}^{\infty} c_k(G)\mu^k = \sum_{k=0}^{|G|} c_k(G)\mu^k,$$

where $c_0(G) = 1$ and $c_k(G)$ $(k \geq 1)$ is the number of directed chains of length k in G.

For any non-empty labeled poset (P,ω), we set $G(P,\omega)$ to be the directed graph $\mathcal{G}(P,\omega)\backslash\{\emptyset, P\}$. Then, for any $k \geq 1$, it is easy to see that $c_k(P,\omega)$ is the same as the number $c_{k-1}(G(P,\omega))$. Hence, by Eq. (2.22) and (2.17), we have the following identities.

Proposition 2.16. *For any non-empty labeled poset* (P,ω), *we have*

$$\begin{aligned}\tilde{e}(P,\omega;\lambda) &= \frac{\lambda}{(1-\lambda)^2}c(G;\frac{\lambda}{1-\lambda})\\ e(P,\omega;\lambda) &= \lambda(1-\lambda)^{|P|-1}c(G;\frac{\lambda}{1-\lambda}).\end{aligned}$$

Theorem 2.17. *Let* (P,ω) *be a non-empty labeled poset. Then we have*

$$e(P,\omega;\lambda) = \lambda \sum_{\emptyset\neq S\in \mathfrak{I}_\omega(P)} (1-\lambda)^{|S|-1} e(P\backslash S,\omega;\lambda) \tag{2.24}$$

and

$$\tilde{e}(P,\omega;\lambda) = \frac{\lambda}{1-\lambda} \sum_{\emptyset\neq S\in \mathfrak{I}_\omega(P)} \tilde{e}(P\backslash S,\omega;\lambda). \tag{2.25}$$

It is easy to see that Eq. (2.24) and (2.25) are equivalent to each other, so we need to prove only one of them.

First Proof: First, from the equation

$$\lambda(I-\lambda\widetilde{A}(P,\omega))^{-1}\widetilde{A}(P,\omega) = (I-\lambda\widetilde{A}(P,\omega))^{-1} - I$$

and Eq. (2.20), we get

$$\lambda\widetilde{A}(P,\omega)E(P,\omega;\lambda) = E(P,\omega;\lambda) - I.$$

Now by comparing the $(\emptyset, P)^{th}$ entries of both sides of the equation above, we have

$$\begin{aligned}\lambda \sum_{S\in \mathfrak{I}_\omega(P)} \tilde{e}(P\backslash S,\omega;\lambda) &= \lambda\tilde{e}(P,\omega,\lambda) + \lambda \sum_{\emptyset\neq S\in \mathfrak{I}_\omega(P)} \tilde{e}(P\backslash S,\omega;\lambda)\\ &= \tilde{e}(P,\omega,\lambda).\end{aligned}$$

By solving for $\tilde{e}(P,\omega,\lambda)$ in the equation above, we get Eq. (2.25). □

We also can prove Eq. (2.24) directly as follows.

Second Proof: First, by Theorem 2.7, we have

$$\Omega(P,\omega;t+1) - \Omega(P,\omega;t) = \sum_{\emptyset\neq S\in \mathfrak{I}_\omega(P)} \Omega(P\backslash S,\omega;t). \tag{2.26}$$

Hence we have

$$\sum_{n=0}^{\infty}(\Omega(P,\omega;n+1) - \Omega(P,\omega;n))\lambda^n = \sum_{\emptyset\neq S\in \mathfrak{I}_\omega(P)} \sum_{n=0}^{\infty} \Omega(P\backslash S,\omega;n)\lambda^n.$$

and since $\Omega(P, \omega; 0) = 0$ (see Proposition 2.10), we have

$$\frac{1-\lambda}{\lambda} \sum_{n=0}^{\infty} \Omega(P, \omega; n) \lambda^n = \sum_{\emptyset \neq S \in \mathcal{J}_\omega(P)} \frac{e(P \backslash S, \omega; \lambda)}{(1-\lambda)^{|P \backslash S|+1}},$$

and

$$\frac{1-\lambda}{\lambda} \frac{e(P, \omega; \lambda)}{(1-\lambda)^{|P|+1}} = \sum_{\emptyset \neq S \subset \mathcal{J}_\omega(P)} \frac{e(P \backslash S, \omega; \lambda)}{(1-\lambda)^{|P \backslash S|+1}}.$$

Hence we have Eq. (2.25). □

Finally, let us apply our recursion formula (2.25) to the anti-chains A_n $(n \geq 1)$. Note that, for any fixed $n \in \mathbb{N}^+$, the labeled poset (A_n, ω) is always a naturally labeled poset for any labeling ω and hence, for fixed n, all (A_n, ω) are equivalent. The Eulerian polynomial $A_n(\lambda) = e(A_n, \omega; \lambda)$ of (A_n, ω) is known as the n^{th} *Eulerian polynomial.*

Corollary 2.18. *For any $n \geq 2$, we have*

$$A_n(\lambda) = \lambda \sum_{k=1}^{n} \binom{n}{k} A_{n-k}(\lambda)(1-\lambda)^{k-1}. \tag{2.27}$$

Proof: This is a direct consequence of Eq. (2.25), because any subset S of $(A_n; \omega)$ is in $\mathcal{J}_\omega(A_n)$ and $A_n \backslash S = A_{n-|S|}$. □

Actually, we have another recursion formula for $A_n(\lambda)$ $(n \in \mathbb{N}^+)$, which is better in the sense that it involves only $A_{n-1}(\lambda)$.

Proposition 2.19. *For any $n \geq 1$, we have*

$$A_n(\lambda) = \lambda(1-\lambda)A'_{n-1}(\lambda) + n\lambda A_{n-1}(\lambda). \tag{2.28}$$

Proof: Note that, for any $n \geq 1$, we have $\Omega(A_n, \omega; t) = t^n$. (This follows from the definition of the order polynomial.) Therefore

$$\frac{A_n(\lambda)}{(1-\lambda)^{n+1}} = \sum_{k=0}^{\infty} k^n \lambda^k = \left(\lambda \frac{d}{d\lambda}\right)^n \frac{1}{1-\lambda}. \tag{2.29}$$

Hence we have

$$\frac{A_n(\lambda)}{(1-\lambda)^{n+1}} = \lambda \frac{d}{d\lambda} \frac{A_{n-1}(\lambda)}{(1-\lambda)^n} = \frac{\lambda(1-\lambda)A'_{n-1}(\lambda) + n\lambda A_{n-1}(\lambda)}{(1-\lambda)^{n+1}}$$

and Eq. (2.28) follows immediately. □

2.3. **Applications to Bernoulli Numbers and Bernoulli Polynomials.** In this subsection, we use the results on order polynomials of labeled posets derived in Subsection 2.2 to give some combinatorial results on Bernoulli numbers and Bernoulli polynomials.

Consider strictly labeled shrubs (S_n, ω) $(n \in \mathbb{N})$, i.e. the labeling $\omega : S_n \to \mathbb{N}^+$ is an order-reversing map. Note that, by Lemma 2.1, the order polynomial $\Omega(S_n, \omega; t)$ is same as the strict order polynomial $\bar{\Omega}(S_n; t)$ of the unlabeled poset S_n. Also note that, the only ω-natural ideals of (S_n, ω) are $\emptyset$ and the set consisting of the unique minimum element of S_n.

First let us recall that the Bernoulli numbers b_n $(n \in \mathbb{N})$ are defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} \tag{2.30}$$

and the Bernoulli polynomials $B_n(t)$ $(n \in \mathbb{N})$ are defined by

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!}. \tag{2.31}$$

Proposition 2.20. [WZ] *For the labeled poset (S_n, ω) $(n \in \mathbb{N}^+)$, we have*

$$\phi(S_n, \omega) = b_n \tag{2.32}$$

and

$$\Omega(S_n, \omega; t) = \int_0^t B_n(u)\, du. \tag{2.33}$$

Two different proofs are given in [WZ]. Here we give yet another one.

Proof: Clearly, we need to prove only Eq. (2.33), for $\phi(S_n, \omega)$ and b_n are the coefficients of t in polynomials $\Omega(S_n, \omega; t)$ and $\int_0^t B_n(u)\, du$, respectively. By applying Eq. (2.6), we have

$$\Delta\Omega(S_n, \omega; t) = \Omega(A_n, \omega; t) = t^n.$$

Since both $\Omega(S_n, \omega; t)$ and $\int_0^t B_n(u)\, du$ have no constant terms, it suffices to show that $\int_0^t B_n(u)\, du$ also satisfies the equation above.

First, from the well-known equation $\Delta B_n(u) = nu^{n-1}$, we have

$$\int_0^t \Delta B_n(u)\, du = t^n.$$

But

$$\begin{aligned}\int_0^t \Delta B_n(u)\,du &= \int_0^t B_n(u+1)\,du - \int_0^t B_n(u)\,du \\ &= \int_1^{t+1} B_n(u)\,du - \int_0^t B_n(u)\,du \\ &= \Delta \int_0^t B_n(u)\,du - \int_0^1 B_n(u)\,du.\end{aligned}$$

Since $\int_0^1 \frac{xe^{tx}}{e^x-1}\,dt = 1$, we have $\int_0^1 B_n(u)\,du = 0$ for any $n \in \mathbb{N}^+$. Therefore, we see that

$$\Delta \int_0^t B_n(u)\,du = t^n.$$

Hence, we have Eq. (2.33). □

Note that Eq. (2.33) provides a combinatorial interpretation for the Bernoulli polynomials $B_n(t)$ ($n \in \mathbb{N}^+$). To consider the combinatorial interpretation of the Bernoulli numbers b_n ($n \in \mathbb{N}^+$), first, from Eq. (2.32) and (2.10), we have

$$b_n = \sum_{k=1}^{n+1} (-1)^{k-1} \frac{c_k(S_n, \omega)}{k}. \tag{2.34}$$

Therefore, b_n is just the weighted alternating sum of the numbers of directed paths of the ω-graph $\mathcal{G}(S_n, \omega)$ of the labeled poset (S_n, ω) connecting $\emptyset$ and S_n, where such a directed path of length k ($k \in \mathbb{N}^+$) is weighted by $\frac{1}{k}$. By counting the numbers $c_k(S_n, \omega)$ ($k \in \mathbb{N}^+$) of directed chains in $\mathcal{G}(S_n, \omega)$ connecting $\emptyset$ and S_n, it is easy to see that

$$c_1(S_n, \omega) = 0; \tag{2.35}$$

$$c_k(S_n, \omega) = \sum_{\substack{n_1+\cdots+n_{k-1}=n \\ n_i \geq 1}} \binom{n}{n_1, \cdots, n_{k-1}} \tag{2.36}$$

for any $k \geq 2$. Therefore, we have

$$b_n = \sum_{k=1}^{n} (-1)^k \frac{1}{k+1} \sum_{\substack{n_1+\cdots+n_k=n \\ n_i \geq 1}} \binom{n}{n_1, \cdots, n_k}. \tag{2.37}$$

On the other hand, from the generating function Eq. (2.31), it is easy to derive that

$$b_n = \sum_{r=1}^{n} (-1)^r n! \sum_{\substack{n_1+\cdots+n_r=n \\ n_i\geq 1}} \frac{1}{(n_1+1)!}\frac{1}{(n_2+1)!}\cdots\frac{1}{(n_r+1)!}. \tag{2.38}$$

Hence, we get the following identity.

Lemma 2.21. *For any $n \geq 1$, we have*

$$\begin{aligned} &\sum_{k=1}^{n} (-1)^k \frac{1}{k+1} \sum_{\substack{n_1+\cdots+n_k=n \\ n_i\geq 1}} \frac{1}{n_1!}\frac{1}{n_2!}\cdots\frac{1}{n_k!} \\ &= \sum_{k=1}^{n} (-1)^k \sum_{\substack{n_1+\cdots+n_k=n \\ n_i\geq 1}} \frac{1}{(n_1+1)!}\frac{1}{(n_2+1)!}\cdots\frac{1}{(n_k+1)!}. \end{aligned} \tag{2.39}$$

The identity above can also be proved directly by the method of generating functions as follows.

2nd Proof : First, consider the generating function

$$\frac{\ln(1+U)}{U} = \sum_{k=0}^{\infty} (-1)^k \frac{U^k}{k+1}. \tag{2.40}$$

Now set $U = e^x - 1$, the right hand side of the equation above gives

$$\begin{aligned} &\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} \left(\sum_{i=1}^{\infty} \frac{x^i}{i} \right)^k \\ = &\sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} (-1)^k \frac{1}{k+1} \sum_{\substack{n_1+\cdots+n_k=n \\ n_i\geq 1}} \frac{1}{n_1!}\frac{1}{n_2!}\cdots\frac{1}{n_k!} \right) x^n. \end{aligned}$$

On the other hand, the left hand side of Eq. (2.40) gives

$$\frac{\ln(1+U)}{U} = \frac{\ln(1+(e^x-1))}{e^x-1} = \frac{x}{e^x-1}. \tag{2.41}$$

Note that

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} (-1)^k \sum_{\substack{n_1+\cdots+n_k=n \\ n_i\geq 1}} \frac{1}{(n_1+1)!}\cdots\frac{1}{(n_k+1)!} \right) x^n.$$

By comparing the coefficients of x^n in Eq. (2.40) and using the expression of b_n in Eq. (2.38), we are done. $\square$

One interesting observation is that, from Eq. (2.37), it is easy to see that $(n+1)!b_n = (n+1)!\phi(S_n, \omega)$ is always a positive integer for any $n \in \mathbb{N}$. Hence we have the following interesting result about Bernoulli numbers.

Corollary 2.22. *The Bernoulli's number b_n $(n \in \mathbb{N})$ can be written as the following* **fraction**

$$b_n = \frac{(n+1)! \sum_{k=1}^{n} (-1)^k \frac{1}{k+1} \sum_{\substack{n_1+\cdots+n_k=n \\ n_i \geq 1}} \binom{n}{n_1, n_2, \cdots, n_k}}{(n+1)!}.$$

In particular, the denominator of b_n in reduced form is always a factor of $(n+1)!$. This fact is not obvious from the expression of b_n given by Eq. (2.38).

3. A New Approach to Order Polynomials of Labeled Posets and Their Generalizations

In this section, motivated by the property of order polynomials of labeled posets given by Theorem 2.7 and the family of invariants for rooted trees constructed in [Z], we introduce a family of invariants for labeled posets. We show that order polynomials $\Omega(P, \omega; t)$ and the invariant $\tilde{e}(P, \omega; \lambda)$ (see Eq. (2.17)) belong to this family of invariants. We also consider the known quasi-symmetric function invariant $K(P, \omega; x)$ of labeled posets (P, ω), which suggests further generalizations of our construction.

3.1. A Family of Invariants of Labeled Posets.

Let A be any vector space over a field k and Ξ a k-linear operator on A. We fix one element $a \in A$ and define an invariant $\Psi(P, \omega)$ for labeled posets (P, ω) by the following algorithm.

Algorithm 3.1. (1) *We first set $\Psi(\emptyset) = a$.*

(2) *For any non-empty labeled poset (P, ω), we define $\Psi(P, \omega)$ recursively by*

$$\Psi(P, \omega) = \Xi \sum_{\emptyset \neq S \in \mathfrak{I}_\omega(P)} \Psi(P \backslash S, \omega) \tag{3.1}$$

The following lemma is obvious.

Lemma 3.2. *Let $\Gamma(P, \omega)$ be an A-valued invariant for labeled posets (P, ω). Then, $\Gamma(P, \omega)$ can be re-defined and calculated by Algorithm 3.1 for some k-linear map Ξ if and only if*

(1) *$\Gamma(\emptyset) = a$, and*

(2) *Γ satisfies Eq. (3.1) for every labeled poset (P, ω).*

Remark 3.3. *In general, the restriction of Algorithm 3.1 to naturally labeled rooted forests is not same as Algorithm* 3.1 *in* [Z], *even when the vector space has an algebra structure. This is because the invariants* $\Psi(P,\omega)$ *defined by Algorithm 3.1 in general do not satisfy the following equation.*

$$\Psi(P_1 \sqcup P_2 \sqcup \cdots \sqcup P_k) = \prod_{i=1}^{k}, \Psi(P_i)$$

where $P_1 \sqcup P_2 \sqcup \cdots \sqcup P_k$ *is the disjoint union of naturally labeled posets* P_i $(1 \leq i \leq k)$. *But, when* $A = \mathbb{C}[t]$, $a = 1$ *and* $\Xi = \Delta^{-1}$, *both algorithms give the order polynomials of rooted forests.* (*See Proposition 3.4 below.*)

3.2. **Re-formulation for order Polynomials and Eulerian Polynomials of Labeled Posets.** From Theorem 2.7 and Lemma 3.2, it is easy to see that the order polynomials $\Omega(P,\omega;t)$ of labeled posets (P,ω) can be reformulated as follows.

Proposition 3.4. *Let* Ψ *be the invariant defined by Algorithm 3.1 with* $A = \mathbb{Q}[t]$, $a = 1$ *and* $\Xi = \Delta^{-1}$. *Then, for any finite poset* P, *we have* $\Psi(P,\omega) = \Omega(P,\omega;t)$.

From Theorem 2.17 and Lemma 3.2, it is easy to see that the invariant $\tilde{e}(P,\omega;\lambda)$ of labeled posets (P,ω) can be reformulated as follows.

Let A be the localization $\mathbb{Q}[\lambda]_{(1-\lambda)}$ of the polynomial algebra $\mathbb{Q}[\lambda]$ at $1-\lambda$, i.e. $A = \mathbb{Q}[\lambda,(1-\lambda)^{-1}]$. Let M_λ be the linear operator of A defined by the multiplication by $\frac{\lambda}{1-\lambda}$. Then we have the following proposition.

Proposition 3.5. *Let* Ψ *be the invariant defined by Algorithm 3.1 with* $A = \mathbb{Q}[\lambda]_{(1-\lambda)}$, $a = \frac{1}{1-\lambda}$ *and* $\Xi = M_\lambda$. *Then, for any labeled poset* (P,ω), *we have* $\Psi(P,\omega) = \tilde{e}(P,\omega;\lambda)$.

Since the Eulerian polynomial $e(P,\omega;\lambda) = (1-\lambda)^{|P|+1}\tilde{e}(P,\omega;\lambda)$ for any labeled posets (P,ω), we see that Eulerian polynomials $e(P,\omega;\lambda)$ can be recovered up to the factor $(1-\lambda)^{|P|+1}$ from an invariant defined Algorithm 3.1.

3.3. **Quasi-Symmetric Functions** K **of Labeled Posets.** First, let us recall that the following well-known quasi-symmetric function invariant $K(P,\omega;x)$ for labeled poset (P,ω).

Let $x = (x_1, x_2, \cdots)$ be a sequence of commutative variables and let $\mathbb{C}[[x]]$ be the formal power series algebra in $\{x_k | k \geq 1\}$ over $\mathbb{C}$. For any labeled poset (P,ω) and any map $\sigma : P \to \mathbb{N}^+$ of sets, we set

$x^\sigma := \prod_{i=1}^\infty x_i^{|\sigma^{-1}(i)|}$ and define

$$K(P,\omega;x) = \sum_\sigma x^\sigma, \tag{3.2}$$

where the sum runs over the set of all ω-order-preserving maps $\sigma : P \to \mathbb{N}^+$.

Recall that an element $f \in \mathbb{C}[[x]]$ is said to be *quasi-symmetric* if the degree of f is bounded, and for any $a_1, a_2, \cdots, a_k \in \mathbb{N}^+$, $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$, the coefficient of the monomial $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ is always same as the coefficient of the monomial $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$. For more general studies on quasi-symmetric functions, see [G], [T], [MR] and [St3].

From Eq. (3.2), it is easy to check that, for any labeled poset (P,ω), $K(P,\omega;x)$ is quasi-symmetric.

Next, let us derive a recursion formula for the quasi-symmetric function invariant $K(P,\omega;x)$ of a labeled poset (P,ω).

Let $S : \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ be the shift operator defined first by setting

$$\begin{aligned} S(1) &= 1; \\ S(x_m) &= x_{m+1} \quad \text{for any } m \geq 2 \end{aligned}$$

and then extending it to the unique $\mathbb{C}$-algebra homomorphism from $\mathbb{C}[[x]]$ to $\mathbb{C}[[x]]$. For any $m \in \mathbb{N}^+$, we define the linear operator Λ_m of $\mathbb{C}[[x]]$ by setting

$$\Lambda_m = \sum_{k=1}^\infty x_k^m S^k \tag{3.3}$$

Proposition 3.6. *For any non-empty labeled poset (P,ω), we have*

$$K(P,\omega;x) = \sum_{\emptyset \neq S \in \mathfrak{I}_\omega(P)} \Lambda_{|S|} K(P \backslash S, \omega; x). \tag{3.4}$$

This proposition is a generalization of Lemma 7.3 in [Z] and the proof is also similar to the proof there.

Proof: Let W be the set of ω-order-preserving maps $\sigma : P \to \mathbb{N}^+$. For each $k \in \mathbb{N}^+$, we set W_k to be the set of $\sigma \in W$ such that $\min(\sigma(P)) = k$. Clearly, W is the disjoint union of the W_k ($k \geq 1$). Also, from the definitions of $K(P,\omega,x)$ and the shift operator S, it is easy to see that, for any labeled poset (P,ω), we have

$$\sum_{\substack{\sigma \in W \\ \sigma(P) \subset \mathbb{N}^+ \backslash [k]}} x^\sigma = S^k K(P,\omega,x). \tag{3.5}$$

Note that, for any $\sigma \in W_k$, $\sigma^{-1}(k)$ is always a non-empty ω-natural ideal, i.e. $\emptyset \neq \sigma^{-1}(k) \in \mathfrak{I}_\omega(P)$. Now let $W(P \backslash S, \omega)$ be the set of all ω-order-preserving maps μ from the labeled poset $(P \backslash S, \omega)$ to $\mathbb{N}^+$ and consider

$$\begin{aligned}
\sum_{\sigma \in W_k} x^\sigma &= \sum_{\emptyset \neq S \in \mathfrak{I}_\omega(P)} \sum_{\substack{\sigma \in W_k \\ \sigma^{-1}(k) = S}} x^\sigma \\
&= \sum_{\emptyset \neq S \in \mathfrak{I}_\omega(P)} x_k^{|S|} \sum_{\substack{\mu \in W(P \backslash S, \omega) \\ \mu(P \backslash S) \subset \mathbb{N}^+ \backslash [k]}} x^\mu \\
&= \sum_{\emptyset \neq S \in \mathfrak{I}_\omega(P)} x_k^{|S|} S^k K(P \backslash S, \omega; x),
\end{aligned}$$

where the last equality follows from Eq. (3.5) and the definition of $K(P, \omega, x)$ (see Eq. (3.2).).

Therefore, we have

$$\begin{aligned}
K(P, \omega; x) &= \sum_{k=1}^{\infty} \sum_{\sigma \in W_k} x^\sigma \\
&= \sum_{k=1}^{\infty} \sum_{\emptyset \neq S \in \mathfrak{I}_\omega(P)} x_k^{|S|} S^k K(P \backslash S, \omega; x) \\
&= \sum_{\emptyset \neq S \in \mathfrak{I}_\omega(P)} \sum_{k=1}^{\infty} x_k^{|S|} S^k K(P \backslash S, \omega; x) \\
&= \sum_{\emptyset \neq S \in \mathfrak{I}_\omega(P)} \Lambda_{|S|} K(P \backslash S, \omega; x).
\end{aligned} \tag{3.6}$$

□

It does not seem possible to recover $K(P, \omega, x)$ from Algorithm 3.1, Rather, the following generalization of Algorithm 3.1 is suggested.

Let A and $a \in A$ as before. Let $\{\Xi_m | m \in \mathbb{N}^+\}$ be a sequence of k-linear operators of A. We define an invariant $\Psi(P, \omega)$ for labeled posets (P, ω) by the following algorithm.

Algorithm 3.7. (1) *We first set* $\Psi(\emptyset) = a$.

(2) *For any labeled poset* P *with* $|P| \geq 2$*, we define* $\Psi(P, \omega)$ *recursively by*

$$\Psi(P, \omega) = \sum_{\emptyset \neq S \in \mathfrak{I}_\omega(P)} \Xi_{|S|} \Psi(P \backslash S, \omega). \tag{3.7}$$

Hence, with $A = \mathbb{C}[[x]]$, $a = 1$ and $\Xi_m = \Lambda_m$ ($m \in \mathbb{N}^+$), the invariant defined by Algorithm 3.7 is same as $K(P, \omega; \lambda)$. From Eq. (2.24) and the fact $e(\emptyset; \lambda) = 1$, we see that the Eulerian polynomials $e(P, \omega; \lambda)$ of labeled posets (P, ω) can also be calculated by Algorithm 3.7 with $A = \mathbb{C}[\lambda]$, $a = 1$ and Ξ_m ($m \in \mathbb{N}^+$) the linear operators of multiplication by $\lambda(1-\lambda)^{m-1}$.

4. A Family Invariants of Unlabeled Posets

In this section, we consider the invariants of unlabeled posets derived from Algorithm 3.1 by identifying unlabeled posets with certain labeled posets.

First, let us identity unlabeled posets with naturally labeled posets. By Lemma 2.1, it is easy to see that the restriction of Algorithm 3.1 on naturally labeled posets gives the following algorithm for order polynomials of unlabeled posets.

Algorithm 4.1. (1) *For the empty poset $\emptyset$, we set $\Psi(\emptyset) = a$.*

(2) *For any non-empty poset P, we define $\Psi(P)$ recursively by*

$$\Psi(P) = \Xi \sum_{\emptyset \neq I \in \mathcal{I}(P)} \Psi(P \backslash S). \tag{4.1}$$

But on the other hand, if we identify unlabeled posets P with labeled posets (P, ω) with order-reversing labelings ω, we get the following algorithm.

Algorithm 4.2. *For the empty poset $\emptyset$, we set $\Psi(\emptyset) = a$.*

(2) *For any non-empty poset P, we define $\Psi(P)$ recursively by*

$$\Psi(P) = \Xi \sum_{\emptyset \neq S \subset L(P)} \Psi(P \backslash S). \tag{4.2}$$

It is easy to see that, with $A = \mathbb{C}[t]$, $a = 1$ and $\Xi = \Delta$; the invariants defined by Algorithm 4.1 and 4.2 are same as order polynomials $\Omega(P)$ and strict order polynomials $\bar{\Omega}(P)$ of unlabeled posets P, respectively. Note that Algorithm 4.2 in general is much more efficient than Algorithm 4.1. Order polynomials $\Omega(P)$ actually can also be calculated by Algorithm 4.2 as follows.

Let ∇ be the $\mathbb{C}$-linear operator defined

$$(\nabla f)(t) = f(t) - f(t-1). \tag{4.3}$$

We also define the operator ∇^{-1} by setting $(\nabla^{-1}f)(t)$ to be the unique polynomial $g(t)$ such that $\nabla f(t) = g(t)$ and $g(0) = 0$. From the definition of strict order polynomials, it is easy to see that

$$\bar{\Omega}(P;1) = \begin{cases} 1 & \text{if } P \text{ is an anti-chain;} \\ 0 & \text{otherwise.} \end{cases}$$

By the following well known identity (see [St2].)

$$\Omega(P;t) = (-1)^{|P|}\bar{\Omega}(P;-t), \tag{4.4}$$

we have

$$\Omega(P;-1) = \begin{cases} (-1)^{|P|} & \text{if } P \text{ is an anti-chain;} \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Proposition 4.3. *For any finite poset P with $|P| \geq 2$, set $\widetilde{\Omega}(P;t) = (-1)^{|P|}\Omega(P;t)$. Then, we have*

$$\widetilde{\Omega}(P;t) = -\nabla^{-1} \sum_{\emptyset \neq S \subset L(P)} \widetilde{\Omega}(P \backslash S, t). \tag{4.6}$$

The proposition above can be proved by using Theorem 2.6 and Eq. (4.5). It also can be proved by using Theorem 2.7 and Eq. (4.4).

Since $\widetilde{\Omega}(\emptyset, t) = 1$, we have

Proposition 4.4. *Let Ψ be the invariant defined by Algorithm 4.2 with $A = \mathbb{C}[t]$, $a = 1$ and $\Xi = -\nabla^{-1}$. Then, for any finite poset P, we have $\Psi_P = \widetilde{\Omega}(P;t) = (-1)^{|P|}\Omega(P;t)$.*

In particular, we see that the order polynomials $\Omega(P;t)$ can also be recovered as an invariant of the type defined by Algorithm 4.2.

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